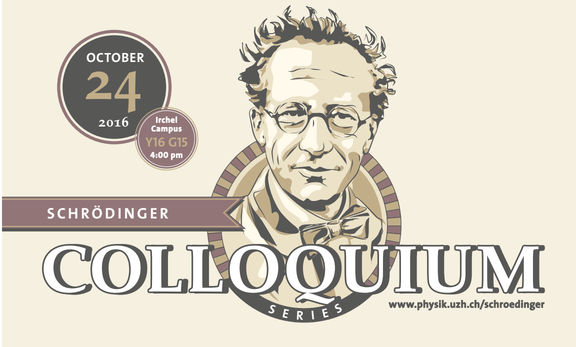
What Quarkonium Taught Me about the Schrödinger Equation, and *vice versa*

Chris Quigg



Erwin Schrödinger Gastprofessur, UniWien, 1991

Birthplace of wave mechanics, 1925–6

Quantization as an Eigenvalue problem $\rightarrow$ *AdP,* 26.01.1926

3. ***Quantisierung als Eigenwertproblem;***
von E. Schrödinger.

(Erste Mitteilung.)

§ 1. In dieser Mitteilung möchte ich zunächst an dem einfachsten Fall des (nichtrelativistischen und ungestörten) Wasserstoffatoms zeigen, daß die übliche Quantisierungsvorschrift sich durch eine andere Forderung ersetzen läßt, in der kein Wort von „ganzen Zahlen" mehr vorkommt. Vielmehr ergibt sich die Ganzzahligkeit auf dieselbe natürliche Art, wie etwa die Ganzzahligkeit der *Knotenzahl* einer schwingenden Saite. Die neue Auffassung ist verallgemeinerungsfähig und rührt, wie ich glaube, sehr tief an das wahre Wesen der Quantenvorschriften.

Die übliche Form der letzteren knüpft an die Hamiltonsche partielle Differentialgleichung an:

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \,. \tag{1}$$

Es wird von dieser Gleichung eine Lösung gesucht, welche sich darstellt als *Summe* von Funktionen je einer einzigen der unabhängigen Variablen q.

Wir führen nun für S eine neue unbekannte ψ ein derart, daß ψ als ein *Produkt* von eingriffigen Funktionen der einzelnen Koordinaten erscheinen würde. D. h. wir setzen

$$S = K \lg \psi \,. \tag{2}$$

Die Konstante K muß aus dimensionellen Gründen eingeführt werden, sie hat die Dimension einer *Wirkung*. Damit erhält man

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E \,. \tag{1'}$$

Wir suchen nun *nicht* eine Lösung der Gleichung (1'), sondern wir stellen folgende Forderung. Gleichung (1') läßt sich bei Vernachlässigung der Massenveränderlichkeit stets, bei Berücksichtigung derselben wenigstens dann, wenn es sich um das *Ein*elektronenproblem handelt, auf die Gestalt bringen: quadratische

$$i\hbar\,\partial\Psi(\mathbf{r},t)/\partial t = \mathcal{H}\Psi(\mathbf{r},t)$$

$$-(\hbar^2/2\mu)\nabla^2\psi(\mathbf{r}) + [V(\mathbf{r}) - E]\,\psi(\mathbf{r}) = 0$$

Materiewellen / Matter waves (1926)

Underlying laws . . . completely known

Quantum Mechanics of Many-Electron Systems.

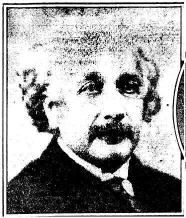
By P. A. M. DIRAC, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received March 12, 1929.)

§ 1. *Introduction.*

The general theory of quantum mechanics is now almost complete, the imperfections that still remain being in connection with the exact fitting in of the theory with relativity ideas. These give rise to difficulties only when high-speed particles are involved, and are therefore of no importance in the consideration of atomic and molecular structure and ordinary chemical reactions, in which it is, indeed, usually sufficiently accurate if one neglects relativity variation of mass with velocity and assumes only Coulomb forces between the various electrons and atomic nuclei. The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble. It therefore becomes desirable that approximate practical methods of applying quantum mechanics should be developed, which can lead to an explanation of the main features of complex atomic systems without too much computation.

“Theories . . . Have Been Exploded”



Times Wide World Photo.
Professor Albert Einstein.

P & A Photo.
Sir Arthur Stanley Eddington.

Times Wide World Photo.
Prince Louis Victor de Broglie.

Dr. Erwin Schrodinger.

Times Wide World Photo.
Dr. Werner Heisenberg.

"God Is a Mathematician Whose Wonders Are Revealed in Symbolic Abstractions and These Are His Prophets."

HOW TO EXPLAIN THE UNIVERSE? SCIENCE IN A QUANDARY

One After Another the Theories Put Out by the Scientists Have Been Exploded and Now Science in Its Uncertainty Has Been Forced to Become Idealistic and to Drop the Idea of a Mechanical Universe

By WALDEMAR KAEMPFFERT.

VICTORIAN science was highly offensive to poets and theologians because it reduced the universe to a machine in which man had no purpose. God was an extraordinarily skillful and far-seeing engineer who dovetailed atoms and stars into a mechanism that could be visualized. It made no difference whether man or his consciousness existed. The cosmic machine kept on moving. Ever since the time of Galileo and Newton the evidence that the universe is a machine had been accumulating at such a rate that it became overwhelming. Ultimately the chief function of science was supposed to be that of predicting the future on the basis of the machine's ceaseless operation.

That mechanical universe is now gone. Science has been forced to become idealistic. In two recent addresses, the one by Sir James Jeans and the other by Sir Arthur Eddington, both leaders in mathematical physics, the strange pass to which exact science has come is summarized. The discovery of the atom's structure, the relation of radiation to matter and the evidence advanced by Einstein and others that our supposedly common-sense views will not bear critical examination have made it necessary to enthrone thought in place of the machine—thought in the form of mathematical concepts. Cause and effect disappear in an exact sense. Chance seems to reign. There is room now for man and his soul. The mathematician has crowded out the old-fashioned philosopher, who is as bewildered by the new physical conception of the universe as if he were the veriest layman. God is a mathematician whose wonders are revealed in symbolic ab-

Matter and energy were one and the same. Man as a collection of electrons was a manifestation of energy. There was nothing really material about him in the accepted sense. For all his solidity he was composed largely of empty space inhabited by occasional bits of energy arranged as atoms. In fact he was relatively about as empty as the solar system. Only the extreme minuteness and vast number of atoms and electrons gave him the aspect and touch of solid flesh.

Because it did not follow Newton's principles obediently, the Rutherford atom came to grief. As they revolved the electrons radiated energy. The Newtonian principles demanded that at the same time they should spiral nearer and nearer the nucleus or "proton." A universe

crude visualization of the wave theory of light.

Planck observed that something entirely different actually occurs. When the one-pound block of wood is dropped ten feet into the pond the other blocks do not bob up and down at all. One block (which one cannot be predicted), suddenly leaps out of the water exactly ten feet. All the energy of the falling block is transmitted to that one floating block without disturbing the other blocks and with nothing happening in the water or the intervening space.

The laws governing these quantum phenomena can be written down, but they are unintelligible at present. The difficulty arises from the way we think and talk of things. The most startling confirmation of Planck's views came from Professor A. H.

moves in a predictable way we deal with probabilities. It is clear that when we deal with probabilities we deal not with a machine but with chance, and with chance only the mathematician can cope. Hence mathematical concepts become more important than mechanism.

This quantum theory was applied by Niels Bohr, another Nobel Prize winner, to explaining the structure and operation of an atom. We have seen that the Rutherford atom was impossible, because atoms would collapse as they radiated and revolved. To account for the fact that the universe was still in existence and showed no signs of collapsing Bohr arbitrarily conceived the electrons as revolving but not radiating. They could revolve forever unless something happened. Whenever the atom radiated light, after having been excited by heat, electric energy or some other force, the electron would jump to the next larger orbit. To rid itself of its acquired energy the electron slipped down to the next inner orbit. This jumping up or down occurred quite in accordance with the quantum theory. By jumping down, an electron lost energy, and the energy lost was a detectable radiation. So many electrons were thus rapidly leaping in a luminous gas or an electric lamp filament that we saw what looked like a continuous glow.

In the Rutherford atom the electrons could revolve in any orbit. In the Bohr atom they revolved only in

The Bohr atom broke down for various reasons. It was objectionable on common-sense grounds. The atom radiated only when an electron jumped. When it did not jump the electron was supposed to revolve quietly—something that could never be observed and therefore verified. It was impossible to tell when an electron would jump or where it would land, so that electrons seemed to be endowed with something that resembled free will. They certainly behaved as if they could make the mathematical calculations necessary to achieve their end.

Young Professor Werner Heisenberg disposed of the Bohr atom by insisting that we have no right in science to go beyond the verifiable. After all, what do we actually know about the atom? Merely that it is composed of electrons and that these radiate under certain conditions. Therefore we must confine ourselves to radiations in our study. In fact, we must give up our picture of the atom as a substantial structure and deal only with effects that can be observed and mathematically interpreted.

The Principle of Uncertainty.

When he thus pins himself and other scientists down to the verifiable, Heisenberg finds reality very different from what science and philosophy suppose. He introduces what is known as the principle of uncertainty or indeterminism, a principle that Eddington believes to be as important as relativity. It may

measurements are reasonably correct only in a very coarse way. They are good enough for workaday purposes. When we deal with atoms and electrons, the very structure of the universe, they are deceptive.

Professor Bridgman asks us to consider the case of an expert billiard player who, by proper manipulation of the cue-ball, makes two balls rebound from the collision as he wishes. The billiard player predicts the future from the present, something possible only with the gross matter of a workaday world. When we come down to atomic realities the case is different. By analogy with the billiard ball we expect the physicist to predict what will happen when a bullet or quantum of radiation collides with an electron.

"The fact is," says Professor

or velocity of the electron I thereby determine whether the electron has position or velocity. The physical properties of the electron are not absolutely inherent in it, but involve also the choice of the observer."

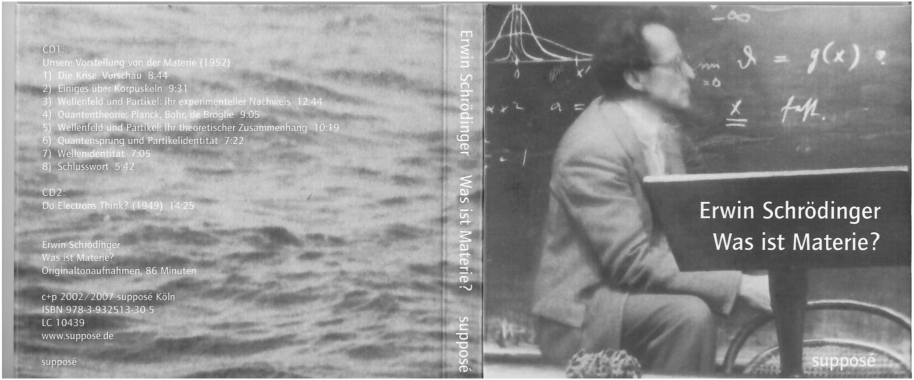
This is much more than mere philosophical quibbling. We are trying to explain reality, the universe, the objects we see and feel. We must of necessity explain them in terms of electrons, the elements of which they are composed. We find ourselves baffled. On an atomic scale we can never identify one electron with another after it has radiated. An electron can be spotted only when it has done something, such as activating a photographic plate, and even then only after the activation.

What happens between the instants when electrons are thus spotted can-

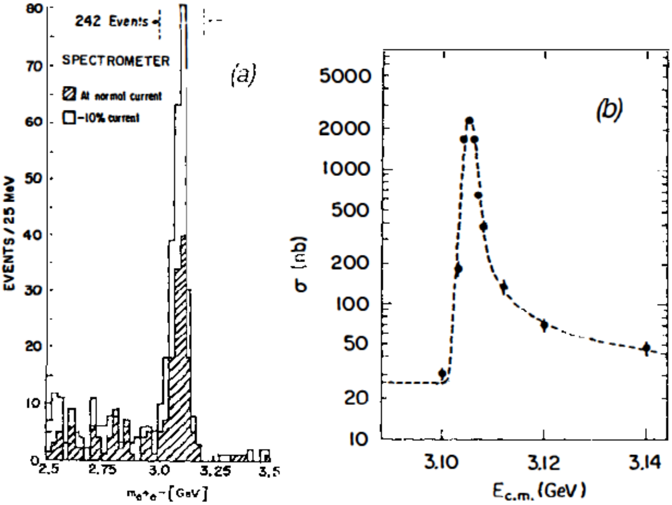
to be composed of quanta or darts when it is studied on a small scale. Consider it on an astronomical scale or in the form of radio (for in radio we deal merely with light that our eyes cannot see) and only the old wave theory can help us in our effort to explain what occurs. Professor Laue proved that X-rays must be light waves by making them form tell-tale diffraction patterns after having passed through a crystal; yet Professor Compton proves by similar means that X-rays are like darts. Electrons are handled as if they were bullets, and they leave tracks on a photographic plate, just the kind of tracks that a bullet would leave. On the other hand Davisson and Germer, following the conclusions of de Broglie and Schrödinger, produce photographic evidence that electrons are only waves of something.

With the ether it is the same. In fact its history, in a sense, is evidence of the steadily increasing bewilderment of science. In its modern form the ether had been postulated to account for the transmission of light. Light was a wave motion—still is, for some purposes. We cannot speak of waves in the ocean without imagining water. So the "luminiferous ether" was invented to explain what waved when a star or a lamp sent us light waves. The ether was so tenuous that it was supposed to pervade everything, even the spaces between atoms of matter. No gas was so rare. Because of its nature it could never be the subject of direct experiment. As new facts were discovered (radio, for example) it became necessary to tinker with the ether. While it was more tenuous than any gas it acquired some of the properties of a jelly. In spite of its diaphanous nature it had to be as

Schrödinger speaks

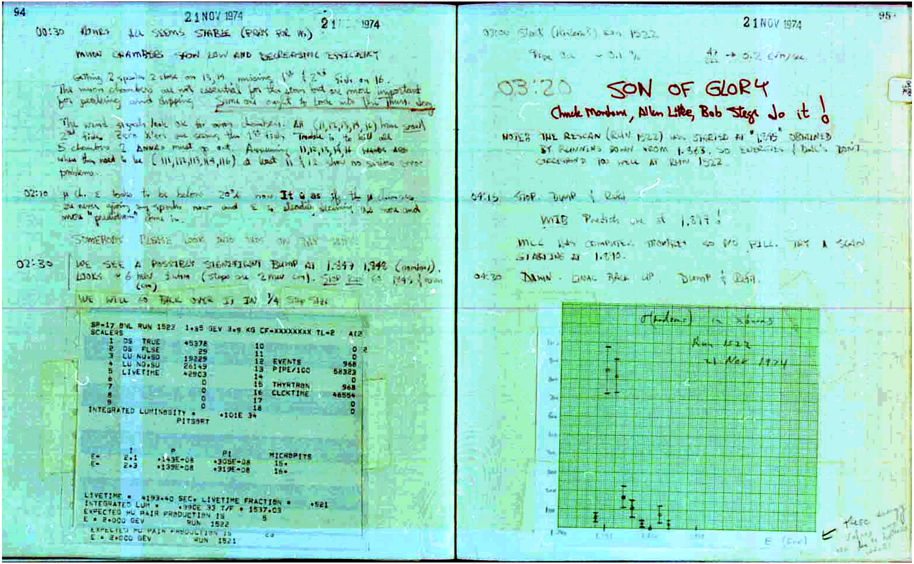


November 11, 1974: J/ψ announced



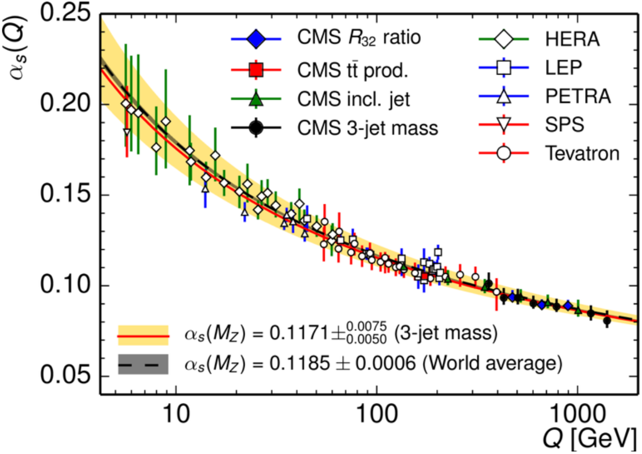
$p + \mathrm{Be} \to e^+e^- + X$ $e^+e^- \to$ hadrons, etc.

November 21, 1974: $e^+e^- \to \psi'$



1973: Perturbation theory for strong interactions?

$$\text{QCD:} \quad 1/\alpha_s(Q) = 1/\alpha_s(\mu) + \frac{33 - 2n_f}{6\pi} \ln(Q/\mu)$$

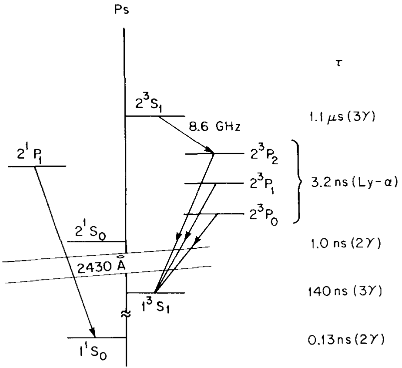


Charmonium ($c\bar{c}$) analogy to Positronium (e^+e^-)

Heavy Quarks and e^+e^- Annihilation*

Thomas Appelquist† and H. David Politzer‡
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 19 November 1974)

The effects of new, heavy quarks are examined in a colored quark-gluon model. The e^+e^- total cross section scales for energies far above any quark mass. However, it is much greater than the scaling prediction in a domain about the nominal two-heavy-quark threshold, despite $\sigma_{e^+e^-}$ being a weak-coupling problem above 2 GeV. We expect spikes at the low end of this domain and a broad enhancement at the upper end.



Charmonium spectroscopy

"Culturally determined" potential $V(r) = -\kappa/r + r/a^2$

Spectroscopy of the New Mesons*

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(Received 11 December 1974)

The interpretation of the narrow boson resonances at 3.1 and 3.7 GeV as charmed quark-antiquark bound states implies the existence of other states. Some of these should be copiously produced in the radiative decays of the 3.7-GeV resonance. We estimate the masses and decay rates of these states and emphasize the importance of γ-ray spectroscopy.

Spectrum of Charmed Quark-Antiquark Bound States*

E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. D. Lane, and T.-M. Yan†
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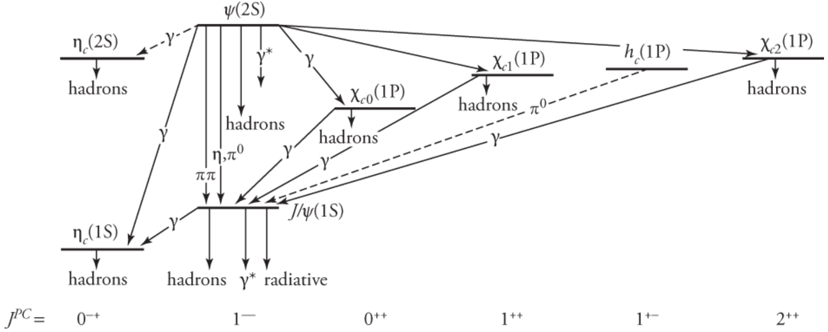
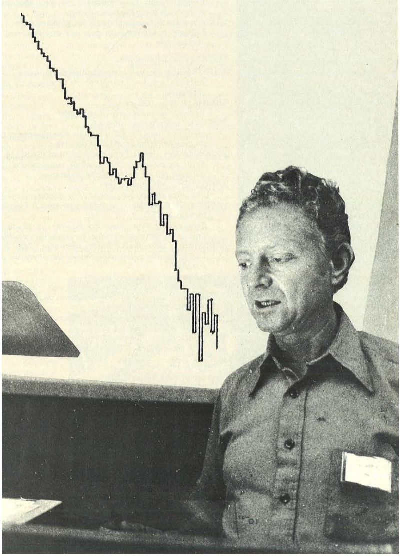
(Received 17 December 1974)

The discovery of narrow resonances at 3.1 and 3.7 GeV and their interpretation as charmed quark-antiquark bound states suggest additional narrow states between 3.0 and 4.3 GeV. A model which incorporates quark confinement is used to determine the quantum numbers and estimate masses and decay widths of these states. Their existence should be revealed by γ-ray transitions among them.

The classic charmonium states

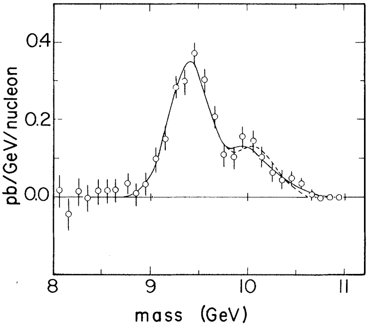


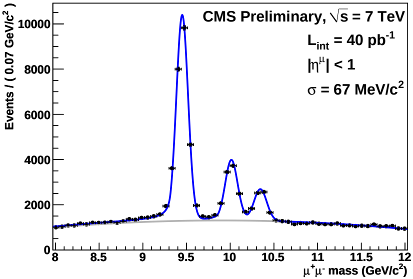
June 16, 1977: Upsilon discovery



$p + (\text{Cu, Pt}) \rightarrow \mu^+\mu^- + \text{anything}$

Υ: From discovery to calibration in dimuons





$\Upsilon' - \Upsilon$ spacing "same as" $\psi' - J/\psi$

E288	$M(\Upsilon') - M(\Upsilon)$	$M(\Upsilon'') - M(\Upsilon)$
Two-level fit	650 ± 30 MeV	
Three-level fit	610 ± 40 MeV	1000 ± 120 MeV
$M(\psi') - M(J/\psi)$	≈ 590 MeV	

Underlying laws not entirely known!

What would ΔE independent of μ imply about interaction?

$$V(r) = C \log r \rightsquigarrow \Delta E \text{ independent of } \mu$$

Schrödinger Equation in 3 dimensions

$$-\frac{\hbar^2}{2\mu}\nabla^2\Psi(\mathbf{r}) + [V(\mathbf{r}) - E]\,\Psi(\mathbf{r}) = 0$$

μ: reduced mass of two-body system

$\mathbf{r}$: relative coordinate

$\Psi(\mathbf{r})$: Schrödinger wave function

$V(\mathbf{r})$: interaction potential

E: energy eigenvalue

For a central potential $V(\mathbf{r}) = V(r)$, separate

$$\Psi(\mathbf{r}) = R(r)Y_{\ell m}(\theta, \phi)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

Radial Equation

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)R(r)-\left[E-V(r)-\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right]R(r)=0$$

Introduce reduced radial wave function $u(r) \equiv rR(r)$:

$$-u''(r)=\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right]u(r)$$

Same form as 1-d Schrödinger eqn., with boundary conditions

$$u(0)=0 \qquad u'(0)=R(0)$$

Wave function normalization:

$$\int d^3\mathbf{r}\,|\Psi(\mathbf{r})|^2=1 \qquad \int_0^\infty dr\,u(r)^2=1$$

Dependence on mass and coupling strength

Schrödinger equation with potential $V(r) = \lambda r^{\nu}$:

$$\frac{\hbar^2}{2\mu} u''(r) + \left[E - \lambda r^{\nu} - \ell(\ell+1)\hbar^2/2\mu r^2\right] u(r) = 0$$

Bring to dimensionless form: $2\mu/\hbar^2$ $\quad [\lambda] = [\hbar^{-\nu}\mu^{1+\nu}]$ $\quad c \equiv 1$

Define scaled measure of length $\rho \equiv (\hbar^2/2\mu\,|\lambda|)^{p} r$, choose p to eliminate dependence on μ and λ, set $w(\rho) \equiv u(r)$

$$w'' = \left[E\left(\frac{\hbar^2}{2\mu\,|\lambda|}\right)^{-2p} - \frac{2\mu\,|\lambda|}{\hbar^2}\,\mathrm{sgn}\,(\lambda)\rho^{\nu}\left(\frac{\hbar^2}{2\mu\,|\lambda|}\right)^{-p(\nu+2)} - \frac{\ell(\ell+1)}{\rho^2}\right] w = 0$$

Dimensions gone if $p = -1/(2+\nu)$, $E = \left(\frac{\hbar^2}{2\mu\,|\lambda|}\right)^{2p} \frac{\hbar^2}{2\mu}\varepsilon$; $[E] = \mu$

$$\boxed{w''(\rho) + [\varepsilon - \mathrm{sgn}\,(\lambda)\rho^{\nu} - \ell(\ell+1)/\rho^2]w(\rho) = 0}$$

Implications for level spacings

$$\Delta E \sim (2\mu/\hbar^2)^{-\nu/(2+\nu)}|\lambda|^{2/(2+\nu)}$$

Potential	Power ν	$\Delta E \sim$
Coulomb	-1	$\mu\,\|\lambda\|^2$
$r^{-1/2}$	$-\frac{1}{2}$	$\mu^{1/3}\|\lambda\|^{4/3}$
Logarithmic	$\to 0$	$C\mu^0$
Linear	1	$\mu^{-1/3}\|\lambda\|^{2/3}$
Harmonic Oscillator	2	$\mu^{-1/2}\|\lambda\|^{1/2}$
∞ Square Well	$\to \infty$	μ^{-1}

Implications for length scales

$$L \sim (2\mu\,|\lambda|/\hbar^2)^{-1/(2+\nu)}$$

Potential	Power ν	$L \sim$
Coulomb	-1	$(\mu\,|\lambda|)^{-1}$
$r^{-1/2}$	$-\frac{1}{2}$	$(\mu\,|\lambda|)^{-2/3}$
Logarithmic	$\to 0$	$(\mu\,|\lambda|)^{-1/2}$
Linear	1	$(\mu\,|\lambda|)^{-1/3}$
Harmonic Oscillator	2	$(\mu\,|\lambda|)^{-1/4}$
∞ Square Well	$\to \infty$	$(\mu\,|\lambda|)^{0}$

The Virial Theorem & Related Theorems

(Two examples of many)

s-wave ($\ell = 0$) wave function at the origin:

$$|\Psi(0)|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle \frac{dV}{dr} \right\rangle$$

Kinetic energy ($\forall\ \ell$):

$$\langle T \rangle = E - \langle V \rangle = \left\langle \frac{r}{2}\frac{dV}{dr} \right\rangle$$

Wave function at the origin

$$-\int_0^\infty dr \quad u'(r)u''(r) = \int_0^\infty dr \quad \frac{2\mu}{\hbar^2}\left[E - V(r)\right] u(r)u'(r)$$

$$u'u'' = \tfrac{1}{2}(u'^2)' \qquad\qquad uu' = \tfrac{1}{2}(u^2)'$$

$$-\left.\frac{u'(r)^2}{2}\right|_0^\infty = \frac{2\mu}{\hbar^2}\left[E - V(r)\right]\left.\frac{u(r)^2}{2}\right|_0^\infty - \frac{1}{2}\frac{2\mu}{\hbar^2}\int_0^\infty dr\left[E - V(r)\right]' u(r)^2$$

Evaluate, simplify:

$$u'(0)^2 = R(0)^2 = 4\pi\left|\Psi(0)\right|^2 = \underbrace{-\frac{2\mu}{\hbar^2}\left[E - V(0)\right]\frac{u(0)^2}{2}}_{0} + \frac{2\mu}{\hbar^2}\left\langle\frac{dV}{dr}\right\rangle$$

$$\boxed{\left|\Psi(0)\right|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle\frac{dV}{dr}\right\rangle}$$

Wave function at the origin: linear potential

$$V(r) = \lambda r$$

$$|\Psi(0)|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle \frac{dV}{dr} \right\rangle \longrightarrow \frac{\mu\lambda}{2\pi\hbar^2}$$

…same for all *s*-waves

Far easier than solving the Schrödinger equation, working out properties of Airy functions!

Virial Theorem

$$-\int_0^\infty dr\, r\, u'(r)u''(r) = \int_0^\infty dr\, r\, \frac{2\mu}{\hbar^2}\left[E - V(r)\right] u(r)u'(r)$$

$$u'u'' = \tfrac{1}{2}(u'^2)' \qquad\qquad uu' = \tfrac{1}{2}(u^2)'$$

LHS:

$$-\tfrac{1}{2}\, \cancel{r u'(r)^2}\Big|_0^{\infty} \xrightarrow{} 0 + \tfrac{1}{2}\int_0^\infty dr\, u'(r)^2 = \tfrac{1}{2}\, \cancel{u(r)u'(r)}\Big|_0^{\infty} \xrightarrow{} 0 = -\tfrac{1}{2}\int_0^\infty dr\, u(r)u''(r)$$

$$\text{Sch. eqn.:}\; -\tfrac{1}{2}\int_0^\infty dr\, \frac{2\mu}{\hbar^2}\left[E - V(r)\right] u(r)^2 = \frac{1}{2}\,\frac{2\mu}{\hbar^2}\langle E - V\rangle$$

$$\text{RHS:}\; \cancel{\frac{2\mu}{\hbar^2}[E - V(r)]\,\frac{u(r)^2}{2}}\Bigg|_0^{\infty} \xrightarrow{} 0 - \frac{1}{2}\,\frac{2\mu}{\hbar^2}\int_0^\infty dr\, \left(r\left[E - V(r)\right]\right)' u(r)^2$$

$$\langle E - V\rangle \equiv \langle T\rangle = \left\langle \frac{r}{2}\,\frac{dV}{dr}\right\rangle$$

Virial theorem, $\langle T \rangle = \langle (r/2) dV/dr \rangle$: special cases

$$\text{Power-law: } V(r) = \lambda r^{\nu}, \quad -2 < \nu < \infty$$

$$\text{Logarithmic: } V(r) = C \ln{(r/r_0)}$$

$$\langle T \rangle \equiv E - \langle V \rangle \equiv \left\langle \frac{r}{2} \nu \lambda r^{\nu - 1} \right\rangle = \frac{\nu}{2} \langle V \rangle = \nu E/(2 + \nu)$$

Examples:

$$\text{Coulomb, } \nu = -1 : \quad \langle T \rangle = -E = -\tfrac{1}{2} \langle V \rangle$$

$$\text{Harmonic oscillator, } \nu = 2 : \quad \langle T \rangle = \langle V \rangle = E/2$$

$$\text{Logarithmic: } \langle T \rangle = \left\langle \frac{r}{2} \frac{C}{r} \right\rangle = C/2$$

General result

Dualities

Connect bound-state spectra of $V(r) = \lambda r^{\nu}$ $(\nu > 0)$
and $\bar{V}(r) = \bar{\lambda} r^{\bar{\nu}}$ $-2 < (\bar{\nu} < 0)$

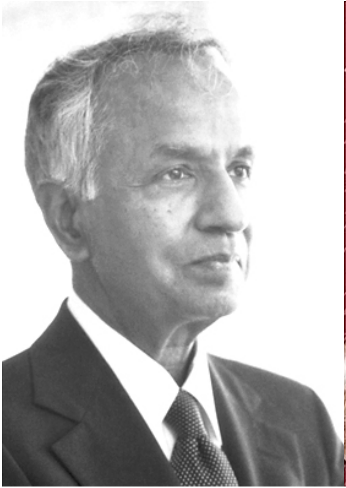
Paired Schrödinger equations

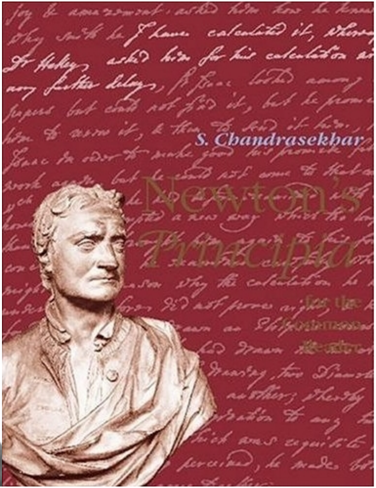
$$\begin{aligned}
\frac{\hbar^2}{2\mu} u''(r) + \left[E - \lambda r^{\nu} - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} \right] u(r) &= 0 \\
\frac{\hbar^2}{2\mu} v''(z) + \left[\bar{E} - \bar{\lambda} z^{\bar{\nu}} - \frac{\bar{\ell}(\bar{\ell}+1)\hbar^2}{2\mu z^2} \right] v(z) &= 0
\end{aligned}$$

$$(\nu + 2)(\bar{\nu} + 2) = 4,\ \bar{E} = \lambda(\bar{\nu}/\nu)^2,\ \bar{\lambda} = -E(\bar{\nu}/\nu)^2,$$
$$z = r^{1+\nu/2},\ (\bar{\ell} + 1/2)^2\nu^2 = (\ell + 1/2)^2\bar{\nu}^2$$

familiar case Coulomb $\Longleftrightarrow$ harmonic oscillator $(\nu, \bar{\nu})$

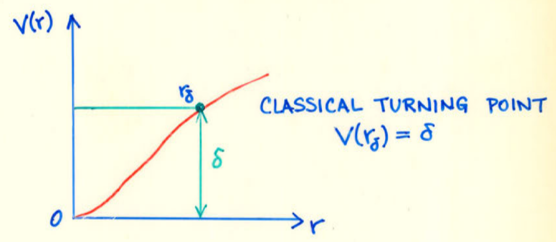
Priority dispute with Isaac Newton





Grant & Rosner, "Classical orbits in power-law potentials"

Number of narrow $^3S_1$ levels $\quad [\delta \equiv 2M(Q\bar{q}) - 2M(Q)]$



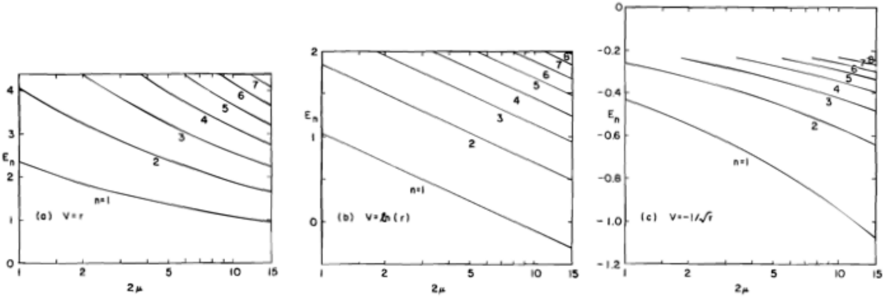
WKB semiclassical quantization condition:

$$\int_0^{r_\delta} dr \underbrace{[2\mu(E - V(r))]^{1/2}}_{\text{local momentum}} = (n - \tfrac{1}{4})\pi\hbar$$

$$\rightsquigarrow (n - \tfrac{1}{4})_{\text{below }\delta} \propto \sqrt{\mu} \text{ for "any" potential}$$

$$\boxed{2 \text{ narrow } \psi \ s\text{-wave levels} \implies 3 \text{ - } 4 \text{ narrow } \Upsilon \ s\text{-wave levels}}$$

How $n \propto \sqrt{\mu}$ is realized for $\Delta E \propto \mu^{(-1/3,0,1/3)}$



$|\Psi_n(0)|^2$ and the level density Evaluating the nonrelativistic connection

$$|\Psi_n(0)|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle \frac{dV}{dr} \right\rangle_n$$

in semiclassical approximation, connect the square of the *s*-wave wave function at the origin to the level density:

$$|\Psi_n(0)|^2 = \frac{(2\mu)^{3/2}}{4\pi^2\hbar^3} E_n^{1/2} \frac{dE_n}{dn}$$

(for a nonsingular potential). Result $\forall\ \ell$

Elementary application

For $V(r) = \lambda r$, $|\Psi_n(0)|^2$: independent of n, $\rightsquigarrow E_n \propto n^{2/3}$.

Reconstructing the potential from the spectrum

Semiclassical Inverse Problem

For a monotonically increasing potential, the semiclassical quantization condition

$$\int_0^{r_0} dr\{2\mu[E - V(r)]\}^{1/2} = (n - \tfrac{1}{4})\pi\hbar$$

connects the shape of the potential to the level density:

$$r(V) = \frac{2\hbar}{(2\mu)^{1/2}} \int_0^V dE(V - E)^{1/2} \left[\frac{dE_n}{dn}\right]^{-1}$$

cf. Gold'man–Krivchenko, *Problems in QM*

Designer potentials

Construct a symmetric, one-dimensional potential
that supports N bound states at specified E_n

...out of reflectionless potentials

$$V(x) = -2\kappa^2 \operatorname{sech}^2[\kappa(x - x_0)]$$

...single bound state at $E = -\kappa^2$

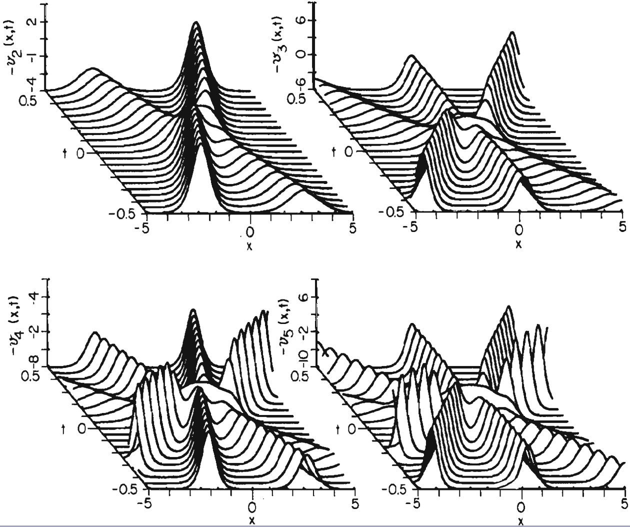
N-level reflectionless potential: N-solitary-wave solution to

$$v_t - 6vv_x + v_{xxx} = 0 \quad \text{(Korteweg–de Vries)}$$

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation." — John Scott Russell

Reflectionless potentials as Korteweg-de Vries solitons: harmonic oscillator example

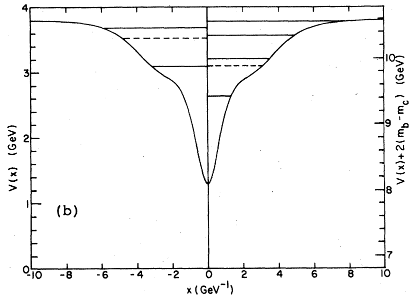
In one dimension, specify energy eigenvalues:



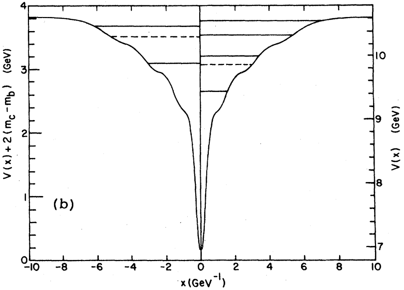
Flavor independence of the $Q\bar{Q}$ interaction

In three dimensions, specify energy eigenvalues (odd-parity) and wave functions at origin (even-parity)

from charmonium



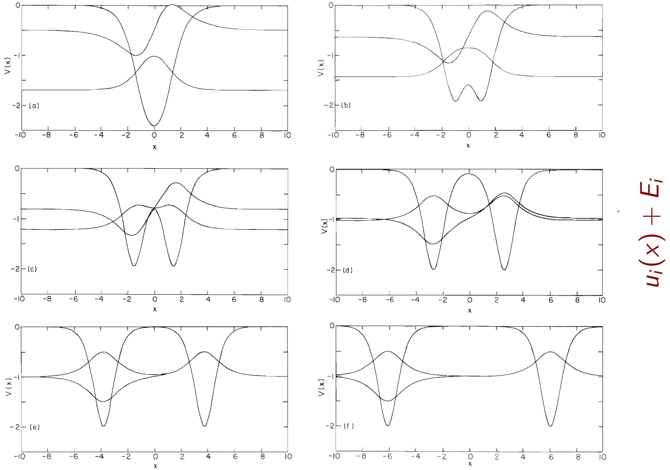
from bottomonium



$(c\bar{c})$ spectrum on left, $(b\bar{b})$ on right in each frame

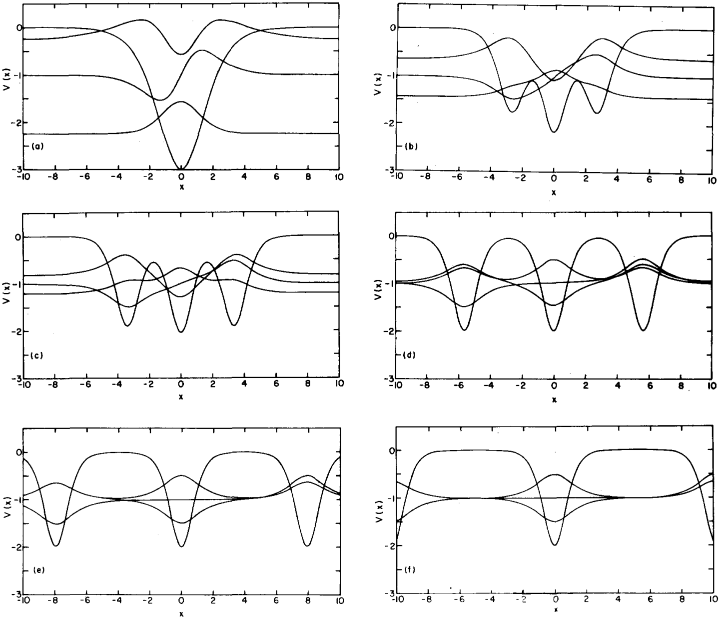
No degenerate levels in one dimensional QM

(simple Wronskian proof, if no pathologies)



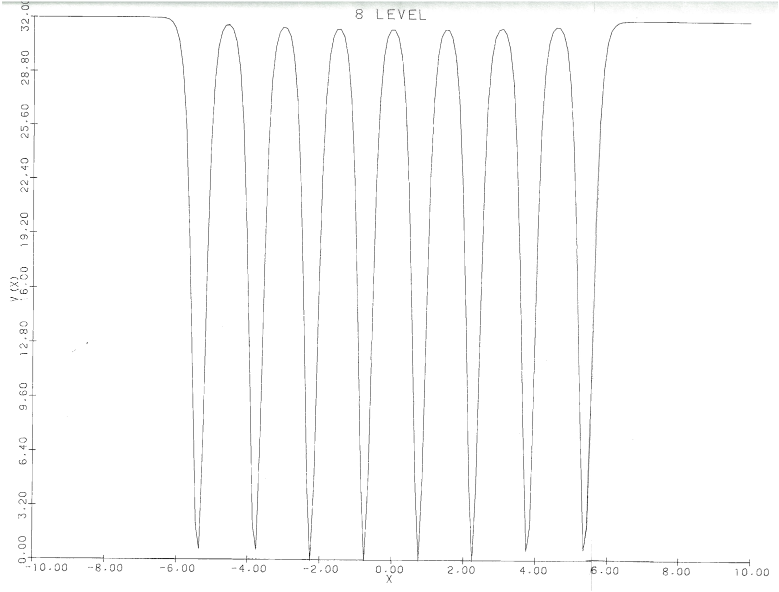
As levels approach, two buckets retreat to $\pm\infty$

Band structure $\rightsquigarrow$ periodic potential: 3 levels

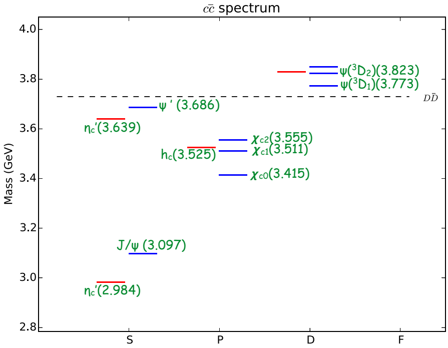


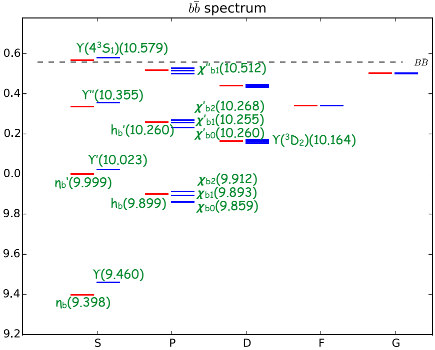
$u_i(x) + E_i$

Band structure ⇝ periodic potential: 8 levels



ψ and Υ narrow levels





- 2 narrow states still unobserved

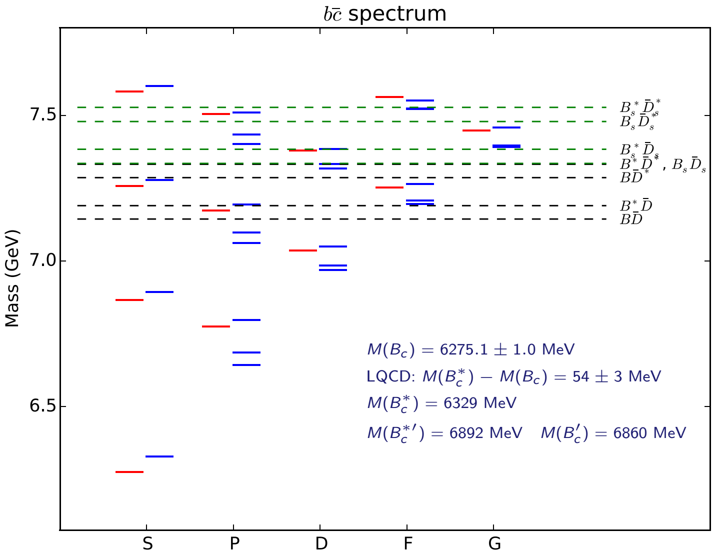
18 narrow states still unobserved

Eichten

$\psi_3(3840)\ {}^3\mathrm{D}_3(J^{PC} = 3^{--}) \to D\bar{D}\ [\pi^+\pi^- J/\psi], \Gamma \lesssim \text{few MeV}$

$\eta_{c2}(3825)\ {}^1\mathrm{D}_2(J^{PC} = 2^{-+}) \to \text{hadrons}, \Gamma \approx 110\ \text{keV}$

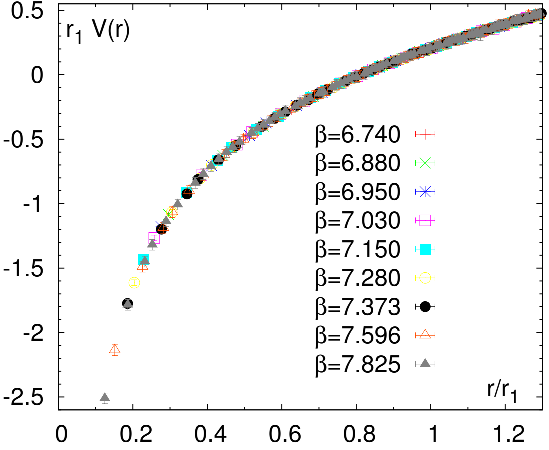
Mesons with beauty and charm



14 narrow states expected below flavor threshold

Eichten / CQ

Static potential from $(2+1)$-flavor lattice QCD



Matches phenomenological determinations

HotQCD Collaboration

Quarkonium-associated states

Table 2: As in Table 1, but for new states near the first open flavor thresholds in the $c\bar{c}$ and $b\bar{b}$ regions, ordered by mass. For $X(3872)$, the values given are based only upon decays to $\pi^+\pi^- J/\psi$. Updated from [7] with kind permission, copyright (2011), Springer, and [8] with kind permission from the authors.

State	m (MeV)	Γ (MeV)	J^{PC}	Process (mode)	Experiment (#σ)	Year	Status
$X(3872)$	3871.68±0.17	< 1.2	1^{++}	$B \to K\,(\pi^+\pi^- J/\psi)$	Belle [42,43] (10.3), BaBar [44] (8.6)	2003	OK
				$p\bar{p} \to (\pi^+\pi^- J/\psi) + ...$	CDF [45–47] (np), D0 [48] (5.2)		
				$B \to K\,(\omega J/\psi)$	Belle [49] (4.3), BaBar [50] (4.9)		
				$B \to K\,(D^{*0}\overline{D}^0)$	Belle [51,52] (6.4), BaBar [53] (4.9)		
				$B \to K\,(\gamma J/\psi)$	Belle [54] (4.0), BaBar [55,56] (3.6), LHCb [57] (>10)		
				$B \to K\,(\gamma\psi(2S))$	BaBar [56] (3.5), Belle [54] (0.4), LHCb [57] (4.4)		
				$pp \to (\pi^+\pi^- J/\psi) + ...$	LHCb [58,59,60] (np)		
$Z_c(3900)$	3891.2 ± 3.3	40 ± 8	1^{+-}	$Y(4260) \to \pi^-(\pi^+ J/\psi)$	BESIII [61](> 8), Belle [62](5.2)	2013	OK
					CLEO data [63](>5)		
				$Y(4260) \to \pi^0(\pi^0 J/\psi)$	BESIII [64](10.4)		
					CLEO data [63](3.5)		
				$Y(4260) \to \pi^-(D\bar{D}^*)^+$	BESIII [65](18)		
				$Y(4260) \to \pi^0(D\bar{D}^*)^0$	BESIII [66](> 10)		
$Z_c(4020)$	4022.9 ± 2.8	7.9 ± 3.7	1^{+-}	$Y(4260, 4360) \to \pi^-(\pi^+ h_c)$	BESIII [67](8.9)	2013	NC!
				$Y(4260, 4360) \to \pi^0(\pi^0 h_c)$	BESIII [68](> 5)		
				$Y(4260) \to \pi^-(D^*\bar{D}^*)^+$	BESIII [69](10)		
				$Y(4260) \to \pi^0(D^*\bar{D}^*)^0$	BESIII [70](5.9)		
$Z_b(10610)$	10607.2 ± 2.0	18.4 ± 2.4	1^{+-}	$\Upsilon(10860) \to \pi^-(\pi^+\Upsilon(1S,2S,3S))$	Belle [71](> 10) [72]	2011	NC!
				$\Upsilon(10860) \to \pi^-(\pi^+ h_b(1P,2P))$	Belle [71](16)		
				$\Upsilon(10860) \to \pi^0(\pi^0\Upsilon(1S,2S,3S))$	Belle [73] (6.5)		
				$\Upsilon(10860) \to \pi^-(B\bar{B}^*)^+$	Belle [74](> 8)		
$Z_b(10650)$	10652.2 ± 1.5	11.5 ± 2.2	1^{+-}	$\Upsilon(10860) \to \pi^-(\pi^+\Upsilon(1S,2S,3S))$	Belle [71](>10)	2011	OK
				$\Upsilon(10860) \to \pi^-(\pi^+ h_b(1P,2P))$	Belle [71](16)		
				$\Upsilon(10860) \to \pi^-(B^*\bar{B}^*)^+$	Belle [74](6.8)		

State	m (MeV)	Γ (MeV)	J^{PC}	Process (mode)	Experiment (#σ)	Year	Status
$X(3915)$	3917.4 ± 2.7	$28^{+10}_{-\ 9}$	$0/2^{++}$	$B \to K\,(\omega J/\psi)$	Belle [75] (8.1), BaBar [50] (np)	2004	OK
				$e^+e^- \to e^+e^-\omega J/\psi$	Belle [76] (7.7), BaBar [77] (19)		
$\chi_{c2}(2P)$	3927.2 ± 2.6	24±6	2^{++}	$e^+e^- \to e^+e^-(D\bar{D})$	Belle [78] (5.3), BaBar [79]	2005	OK
$X(3940)$	3942^{+9}_{-8}	37^{+27}_{-17}	$?^{?+}$	$e^+e^- \to J/\psi\,(D\overline{D}^*)$	Belle [80] (6.0)	2007	NC!
				$e^+e^- \to J/\psi\,(...)$	Belle [22] (5.0)		
$Y(4008)$	$4008^{+121}_{-\ 49}$	226±97	1^{--}	$e^+e^- \to \gamma(\pi^+\pi^- J/\psi)$	Belle [81] (7.4)	2007	NC!
$Z_1(4050)^+$	4051^{+24}_{-43}	82^{+51}_{-55}	?	$B \to K\,(\pi^+\chi_{c1}(1P))$	Belle [82] (5.0), BaBar [83] (1.1)	2008	NC!
$Y(4140)$	4145.8 ± 2.6	18 ± 8	$?^{?+}$	$B^+ \to K^+(\phi J/\psi)$	CDF [84,85] (5.0)	2009	NC!
					D0 [86] (3.1), CMS [87] (>5)		
					Belle [88] (1.9), LHCb [89] (1.4), BaBar [90]		
				$e^+e^- \to e^+e^-\,(\phi J/\psi)$	Belle [91] (3.2)	2009	NC!
$X(4160)$	4156^{+29}_{-25}	139^{+113}_{-65}	$?^{?+}$	$e^+e^- \to J/\psi\,(D\overline{D}^*)$	Belle [80] (5.5)	2007	NC!
$Z_c(4200)^+$	4196^{+35}_{-32}	370^{+99}_{-149}	1^+	$\bar{B}^0 \to K^-(J/\psi\pi^+)$	Belle [92] (6.2)	2014	NC!
$Z_2(4250)^+$	$4248^{+185}_{-\ 45}$	$177^{+321}_{-\ 72}$	?	$B \to K\,(\pi^+\chi_{c1}(1P))$	Belle [82] (5.0), BaBar [83] (2.0)	2008	NC!
$Y(4260)$	4263^{+8}_{-9}	95±14	1^{--}	$e^+e^- \to \gamma\,(\pi^+\pi^- J/\psi)$	BaBar [93,94] (8.0)	2005	OK
					CLEO [95] (5.4), Belle [81] (15)		
				$e^+e^- \to (\pi^+\pi^- J/\psi)$	CLEO [96] (11)		
				$e^+e^- \to (\pi^0\pi^0 J/\psi)$	CLEO [96] (5.1)		
				$e^+e^- \to (f_0(980) J/\psi)$	BaBar [97] (np), Belle [62] (np)		
				$e^+e^- \to (\pi^- Z_c(3900)^+)$	BESIII [61] (8), Belle [62] (5.2)		
				$e^+e^- \to (\gamma\,X(3872))$	BESIII [98] (5.3)		
$Y(4274)$	4293 ± 20	35 ± 16	$?^{?+}$	$B^+ \to K^+(\phi J/\psi)$	CDF [85] (3.1), LHCb [89] (1.0), CMS [87] (>3), D0 [86] (np)	2011	NC!
$X(4350)$	$4350.6^{+4.6}_{-5.1}$	$13.3^{+18.4}_{-10.0}$	$0/2^{++}$	$e^+e^- \to e^+e^-\,(\phi J/\psi)$	Belle [91] (3.2)	2009	NC!
$Y(4360)$	4361 ± 13	74±18	1^{--}	$e^+e^- \to \gamma\,(\pi^+\pi^-\psi(2S))$	BaBar [99,100] (np), Belle [101,102] (8.0)	2007	OK
$Z(4430)^+$	4458 ± 15	166^{+37}_{-32}	1^+	$\bar{B}^0 \to K^-(\pi^+\psi(2S))$	Belle [103,104,105] (6.4), BaBar [106] (2.4), LHCb [107] (13.9)	2007	OK
				$\bar{B}^o \to (J/\psi\pi^+)K^-$	Belle [92] (4.0)		
$X(4630)$	$4634^{+\ 9}_{-11}$	92^{+41}_{-32}	1^{--}	$e^+e^- \to \gamma\,(\Lambda_c^+\Lambda_c^-)$	Belle [108] (8.2)	2007	NC!
$Y(4660)$	4664±12	48±15	1^{--}	$e^+e^- \to \gamma\,(\pi^+\pi^-\psi(2S))$	Belle [101,102] (5.8),BaBar [100] (np)	2007	NC!
$\Upsilon(10860)$	10876 ± 11	55 ± 28	1^{--}	$e^+e^- \to (B^{(*)}_{(s)}\bar{B}^{(*)}_{(s)}(\pi))$	PDG [109] (> 10)	1985	OK
				$e^+e^- \to (\pi\pi\Upsilon(1S,2S,3S))$	Belle [110,71,73,111] (>10)		
				$e^+e^- \to (f_0(980)\Upsilon(1S))$	Belle [71,73] (>5)		
				$e^+e^- \to (\pi Z_b(10610, 10650))$	Belle [71,73] (>10)		
				$e^+e^- \to (\eta\Upsilon(1S,2S))$	Belle [33] (10)		
				$e^+e^- \to (\pi^+\pi^-\Upsilon(1D))$	Belle [112] (9)		
				$e^+e^- \to (\pi^+\pi^- h_b(1P,2P))$	Belle [113] (9)		
$\Upsilon(11020)$	$10987.5^{+11.1}_{-3.3}$	$61.0^{+9.2}_{-27.7}$	1^{--}	$e^+e^- \to (B^{(*)}_{(s)}\bar{B}^{(*)}_{(s)}(\pi))$	PDG [109] (> 10)	1985	OK
				$e^+e^- \to (\pi\pi\Upsilon(1S,2S,3S))$	[111] (>10)		
				$e^+e^- \to (\pi^+\pi^- h_b(1P,2P))$	Belle [113] (9)		

Table 3: As in Table 1, but for new states above the first open flavor thresholds in the $c\bar{c}$ and $b\bar{b}$ regions, ordered by mass. $X(3945)$ and $Y(3940)$ have been subsumed under $X(3940)$ due to compatible properties. The $\chi_{c0}(3915)$ is now changed back to $X(3915)$ as explained in the main text. The state known as $Z(3930)$ appears as the $\chi_{c2}(2P)$ in Table 1. In some cases experiment still allows two J^{PC} values, in which case both appear. See also the reviews in [1–8].

Quarkonium ⇔ Schrödinger Equation

Using nonrelativistic quantum mechanics embodied in the Schrödinger Equation, we (work of many hands) have . . .

Made a template for the $c\bar{c}$ states: 1P levels as key test

Shown flavor-independence of the Q-$\bar{Q}$ interaction

Characterized the form of the Q-$\bar{Q}$ interaction

Determined b-quark charge, before B-meson discovery

Created a predictive Quarkonium spectroscopy

Probed Lorentz structure of the confining potential

Built a bridge to quantitative lattice-QCD spectroscopy

Established E1, M1, hadronic transition systematics

Predicted B_c ground state

Adapted and generalized the classic sum rules

Constructed framework for analyzing new "exotic" states

Quarkonium ⇔ Schrödinger Equation

Pursuing questions raised by the existence of the $c\bar{c}$ and $b\bar{b}$ families, with $m_b/m_c \approx 3$ - 4, we have

Derived many results either new or forgotten for 40 years

Deduced scaling laws (power-law potentials)

Exploited, generalized virial theorem, etc.

Related bound states of singular & confining potentials

Counted narrow levels, semiclassically

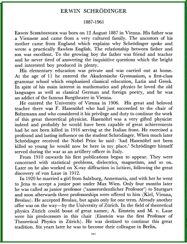
Connected $|\Psi(0)|^2$ with level density

Built up phenomenological potentials from KdV solitons

Gained fresh insights into band structure and periodicity

Learned lessons they don't teach you in school!

More information



ERWIN SCHRÖDINGER

1887-1961

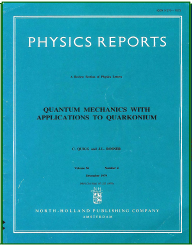
Erwin Schrödinger was born on 12 August 1887 in Vienna. His father was a Viennese and came from a very cultured family. The ancestors of his mother came from England which explains why Schrödinger spoke and wrote a practically flawless English. The relationship between father and son was excellent. To the growing boy the father was friend and teacher and he never tired of answering the inquisitive questions which the bright and interested boy produced in plenty.

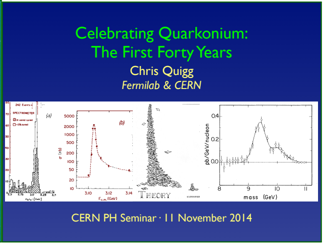
His elementary education was private and was carried out at home. At the age of 11 he entered the Akademische Gymnasium, a first-class grammar school which emphasized classical education, Latin and Greek. In spite of his main interest in mathematics and physics he loved the old languages as well as classical German and foreign poetry, and he was an addict of the famous Burgtheater in Vienna.

He entered the University of Vienna in 1906. His great and beloved teacher there was F. Hasenöhrl who had just succeeded to the chair of Boltzmann and who considered it his privilege and duty to continue the work of this great theoretical physicist. Hasenöhrl was a very gifted physicist indeed and probably he would have been capable of great achievements had he not been killed in 1916 serving at the Italian front. He exercised a profound and lasting influence on the student Schrödinger. When much later Schrödinger received the Nobel Prize he said: 'had Hasenöhrl not been killed so young he would now be here in my place.' Schrödinger himself served during the war as an artillery officer in Italy.

From 1910 onwards his first publications began to appear. They were concerned with statistical problems, dielectrics, magnetism, and so on. Later on he also worked on X-ray diffraction in lattices, following the great discovery of von Laue in 1912.

In 1920 he married a girl from Salzburg, Annemaria, and with her he went to Jena to accept a junior post under Max Wien. Only four months later he was called as junior professor ('ausserordentlicher Professor') to Stuttgart and soon afterwards three professorships were offered to him (Kiel, Vienna, Breslau). He accepted Breslau, but again only for one term. Already another offer was on the way—by the University of Zürich. In the field of theoretical physics Zürich could boast of great names; A. Einstein and M. v. Laue were his predecessors in this chair (Einstein was the first Professor of Theoretical Physics in Zürich). He was destined to continue this great tradition. Six years later he was to become their colleague in Berlin.





Thanks to my collaborators: Estia Eichten, Jonathan Rosner, Hank Thacker, Waikwok Kwong, Ken Lane, Peter Moxhay, Jonathan Schonfeld, and to

Supplementary slides . . .

Generalized virial theorems

For general values of ℓ, write

$$-u''(r) = \mathcal{L}u(r),$$

$$\text{with } \mathcal{L} \equiv (2\mu/\hbar^2)\left[E - V(r) - \ell(\ell+1)\hbar^2/2\mu r^2\right]$$

Apply $\int_0^\infty dr\, r^q u'(r)$. Noting that $u_\ell(r) = a_\ell r^{\ell+1}$ as $r \to 0$, we discover

$$(2\ell+1)^2 a_\ell^2 \delta_{q,-2\ell} = -\left\langle 2qr^{q-1}\mathcal{L} + r^q\mathcal{L}' + \tfrac{1}{2}q(q-1)(q-2)r^{q-3}\right\rangle$$

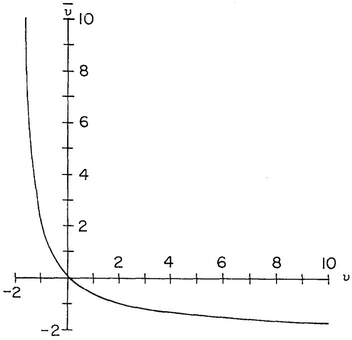
for $q \geq -2\ell$.

Coulomb potential, $V(r) = -|\lambda|\, r^{-1}$:

$$\langle r\rangle = \frac{3\lambda}{4E} + \frac{\ell(\ell+1)\hbar^2}{2\mu\,|\lambda|} = \frac{3n^2 + \ell(\ell+1)\hbar^2}{2\mu\,|\lambda|}$$

Back

Dual power-law potentials ($V = \lambda r^{\nu}, \bar{V} = \bar{\lambda} r^{\bar{\nu}}$)



Back

a_{nl}^2 and level density $\forall \ell$

$$a_{nl}^2 = \frac{(2\mu E_{n\ell}/\hbar^2)^{\ell+\frac{1}{2}}}{\pi[(2\ell+1)!!]^2} \frac{\partial(2\mu E_{n\ell}/\hbar^2)}{\partial n}$$

Bell & Pasupathy

Back